7 Quantum-state reconstruction

The state of a quantum object is commonly described by a normalized Hilbert-space vector $|\Psi\rangle$ or, more generally, by a density operator $\hat{\varrho}$ which is a Hermitian and non-negative valued Hilbert-space operator of trace one. The Hilbert space of the object is usually spanned up by an orthonormalized set of basic vectors $|A\rangle$ representing the eigenvectors (eigenstates) of Hermitian operators $\hat{A}$ associated with a complete set of simultaneously measurable observables (physical quantities) of the object.[1] The eigenvalues A of these operators are the values of the observables which can be registered in a measurement. Here we must distinguish between an individual (single) and an ensemble measurement (i. e., in principle, an infinitely large number of repeated measurements on identically prepared objects). Performing a single measurement on the object, a totally unpredictable value A is observed, in general, and the state of the object has collapsed to the state $|A\rangle$ according to the von Neumann's projection definition of a measurement [von Neumann (1932)]. If the same measurement is repeated immediately after the first measurement (on the same object), the result is now very predictable – the same value A as in the first measurement, is observed. Obviously, owing to the first measurement, the object has been prepared in the state $|A\rangle$. Repeating the measurement many times on an identically prepared object, the relative rate at which the result A is observed approaches the diagonal density-matrix element $\langle A|\hat{\varrho}|A\rangle$ as the number of measurements tends to infinity. Measuring $\langle A|\hat{\varrho}|A\rangle$ for all values of A, the statistics of $\hat{A}$ (and of any function of $\hat{A}$) are known. To completely describe the quantum state, i. e., to determine all the quantum-statistical properties of the object, knowledge of all density-matrix elements $\langle A|\hat{\varrho}|A'\rangle$ is needed. In particular, the off-diagonal elements essentially determine the statistics of such sets of observables $\hat{B}$ that are not compatible with $\hat{A}$ ($[\hat{A},\hat{B}] \neq 0$) and cannot be measured simultaneously with $\hat{A}$. Obviously, the statistics of $\hat{B}$ can also be obtained directly – similar to the statistics of $\hat{A}$ – from an ensemble measurement yielding the diagonal density-matrix elements $\langle B|\hat{\varrho}|B\rangle$ in the basis of the eigenvectors $|B\rangle$ of $\hat{B}$. Now one can

1) For notational convenience we write $\hat{A}$, without further specifying the quantities belonging to the set.

Quantum Optics, Third revised and extended edition. Werner Vogel and Dirk-Gunnar Welsch

ISBN: 3-527-40507-0

proceed to consider other sets of observables which are not compatible with $\hat{A}$ and $\hat{B}$. When a set of observables $\hat{A}_i$, also called a *quorum*, is found, such that the density operator can be represented in the form of [2]

$$\hat{\varrho} = \sum_i \hat{C}_i \, \langle \hat{A}_i \rangle, \tag{7.1}$$

then, in principle, all knowable information on the quantum state of the system can be obtained [Fano (1957)]. In particular, when in a chosen basis $|\Lambda\rangle$ the matrix elements $\langle\Lambda|\hat{C}_i|\Lambda'\rangle$ are bounded for all values of i, then the density-matrix elements $\langle\Lambda|\hat{\varrho}|\Lambda'\rangle$ can be sampled directly from the measured $\langle\hat{A}_i\rangle$ according to the relation

$$\langle\Lambda|\hat{\varrho}|\Lambda'\rangle = \sum_i \langle\Lambda|\hat{C}_i|\Lambda'\rangle\langle\hat{A}_i\rangle. \tag{7.2}$$

Roughly speaking, there have been two routes used to collect measurable data for reconstructing the quantum state of an object under consideration. In the first, which closely follows the method given above, a succession of (ensemble) measurements is made such that a set of noncommutative object observables is measured which carries the complete information about the quantum state. A typical example is the reconstruction of the quantum state from the [in balanced four-port homodyning (Section 6.5.4) measurable] probability distributions $p(x,\varphi) = \langle x,\varphi|\hat{\varrho}|x,\varphi\rangle$ of the phase-rotated quadratures $\hat{x}(\varphi)$ [Eq. (3.143)]. Here the projectors $|x,\varphi\rangle\langle x,\varphi|$ for all phases φ within a π interval play the role of the $\hat{A}_i$ in Eqs (7.1) (Section 7.1.1).

In the second, the object is coupled to a reference system (whose quantum state is well known) such that the measurement of observables of the composite system corresponds to "simultaneous" measurement of noncommutative observables of the object. In this case the number of observables (of the composite system) which must be measured in a succession of (ensemble) measurements can be reduced drastically, but at the expense of the image sharpness of the object. As a result of the additional noise introduced by the reference system, only fuzzy measurements on the object can be performed, which just makes a "simultaneous" measurement of incompatible object observables feasible. A typical example is the [in balanced eight-port homodyning (Section 6.5.5) measurable] Q function, which is given by the diagonal density-matrix elements in the coherent-state basis, $Q(\alpha) = \pi^{-1}\langle\alpha|\hat{\varrho}|\alpha\rangle$ [Eq. (4.67)]. Here, the POVM operators $\pi^{-1}|\alpha\rangle\langle\alpha|$ play the role of the $\hat{A}_i$ in Eqs (7.1) [Eq. (4.83) for $s=-1$]. The Q function can already be obtained from one ensemble measurement of the complex amplitude α in balanced eight-port homodyning, which

2) Here we have assumed a discrete set of observables. For continuous sets of observables, the sum in Eq. (7.1) must be replaced by an integral.

corresponds to a fuzzy measurement of the "joint" probability distribution of two canonically conjugated observables $\hat{x}(\varphi)$ and $\hat{x}(\varphi + \pi/2)$ of the object.

Since the sets of quantities measured via the one or the other route (or an appropriate combination of both) carry the complete information on the quantum state of the object, they can be regarded, in a sense, as representations of the quantum state, which can be more or less close to familiar quantum-state representations, such as the density matrix in the number-state basis or a phase-space function. In any case, the question arises of how to reconstruct, from the measured data, specific quantum-state representations (or specific quantum-statistical properties of the object for which a direct measurement scheme is not available). Again, there have been two typical methods of solving the problem. In the first, equations that relate the measured quantities to the desired quantities are derived and solved either analytically or numerically in order to obtain the desired quantities in terms of the measured quantities. In practice, the measured data are often incomplete, i. e., not all quantities needed for a precise reconstruction are measured,[3] and moreover, the measured data are inaccurate. Obviously, any experiment can only run for a finite time, which prevents one, in principle, from performing an infinite number of repeated measurements in order to obtain precise expectation values. These inadequacies give rise to systematic and statistical errors of the reconstructed quantities, which can be quantified in terms of confidence intervals.

In the second approach, statistical methods are used from the very beginning in order to obtain the best *a posteriori* estimation of the desired quantities on the basis of the available (i. e., incomplete and/or inaccurate) data measured. However, the price to pay may be high. Whereas in the first concept linear equations are typically to be handled and estimates of the desired quantities (including statistical errors) can often be directly sampled from the measured data, application of purely statistical methods, such as the principle of maximum entropy or Bayesian inference, require the treatment of nonlinear equations and a reconstruction in real time is, in general, impossible. Here we concentrate on the first concept, restricting our attention to single-mode systems [for a review on quantum-state reconstruction, see Welsch, Vogel and Opatrný (1999)].

7.1
Optical homodyne tomography

From Section 6.5.4 we know that, in a succession of (phase-shifted) balanced four-port homodyne measurements, the phase-rotated quadrature distribu-

3) To compensate for incomplete data, some *a priori* knowledge of the quantum state is needed, in general.

tion $p(x, \varphi)$ of a signal-field mode can be obtained for various values of the phase parameter φ, provided that 100% detection efficiency is realized. As we will see, the quantum state is then known when $p(x, \varphi)$ is known for all values of φ within a π interval [Vogel and Risken (1989)]. That is to say, all quantum-statistical properties can be obtained from the quadrature-component distributions measured in a π interval.

7.1.1 Quantum state and phase-rotated quadratures

In order to relate the phase-rotated quadrature distributions

$$p(x, \varphi) = \langle x, \varphi|\hat{\varrho}|x, \varphi\rangle = \mathrm{Tr}\,(\hat{\varrho}|x, \varphi\rangle\langle x, \varphi|) \tag{7.3}$$

to the quantum state, we note that the projector

$$|x, \varphi\rangle\langle x, \varphi| = \hat{\delta}[\hat{x}(\varphi) - x] = \frac{1}{2\pi}\int \mathrm{d}y\, e^{-iyx} e^{iy\hat{x}(\varphi)} \tag{7.4}$$

can be expressed in terms of the displacement operator $\hat{D}$ [Eq. (3.44)] as follows:

$$|x, \varphi\rangle\langle x, \varphi| = \frac{1}{2\pi}\int \mathrm{d}y\, e^{-iyx}\hat{D}\big(iye^{-i\varphi}\big). \tag{7.5}$$

We now introduce the characteristic (generating) function $\Psi(y, \varphi)$ of the probability distribution $p(x, \varphi)$ via the Fourier transform

$$p(x, \varphi) = \frac{1}{2\pi}\int \mathrm{d}y\, e^{-iyx}\Psi(y, \varphi), \tag{7.6}$$

$$\Psi(y, \varphi) = \int \mathrm{d}x\, e^{iyx} p(x, \varphi). \tag{7.7}$$

Recalling the definition (3.143) of $\hat{x}(\varphi)$, it is not difficult to prove the following symmetry relations

$$p(-x, \varphi \pm \pi) = p(x, \varphi), \tag{7.8}$$

$$\Psi(-y, \varphi \pm \pi) = \Psi(y, \varphi). \tag{7.9}$$

Substituting the expression (7.4) into Eq. (7.3) and comparing with Eq. (7.6), we can easily see that $\Psi(y, \varphi)$ is the expectation value of a displacement operator:

$$\Psi(y, \varphi) = \mathrm{Tr}\,\{\hat{\varrho}\exp[iy\hat{x}(\varphi)]\} = \mathrm{Tr}\big[\hat{\varrho}\hat{D}\big(iye^{-i\varphi}\big)\big]. \tag{7.10}$$

On the other hand, the characteristic function $\Phi(\alpha; s)$ of an s-parameterized phase-space function $P(\alpha; s)$ is given by

$$\Phi(\alpha; s) = \mathrm{Tr}\,[\hat{\varrho}\hat{D}(\alpha; s)] = e^{\frac{1}{2}s|\alpha|^2}\,\mathrm{Tr}\,[\hat{\varrho}\hat{D}(\alpha)] \tag{7.11}$$

[Eq. (4.91) together with Eqs (4.90) and (4.47)]. Comparing Eqs (7.10) and (7.11), we find that

$$\Psi(y,\varphi) = e^{-\frac{1}{2}sy^2}\Phi\big(iye^{-i\varphi};s\big). \tag{7.12}$$

As expected, $\Psi(y,\varphi)$ can be directly determined from $\Phi(\alpha;s)$. To determine $\Phi(\alpha;s)$ from $\Psi(y,\varphi)$, we note that the relation $\alpha = |\alpha|e^{i\varphi_\alpha} = iye^{-i\varphi}$, with real y, implies that

$$|\alpha|e^{in\pi} = y, \qquad \varphi_\alpha + \varphi = \tfrac{1}{2}(2n+1)\pi. \tag{7.13}$$

From Eqs (7.12) and (7.13) together with the symmetry relation (7.9), it then follows that for chosen values of α and s, $\Phi(\alpha;s)$ can be obtained from $\Psi(y,\varphi)$ according to

$$\Phi(\alpha;s) = e^{\frac{1}{2}s|\alpha|^2}\Psi\big(|\alpha|, \tfrac{1}{2}\pi - \varphi_\alpha\big). \tag{7.14}$$

We can see that for a fixed value of the quadrature phase φ the function $\Phi(\alpha;s)$ can be determined from $\Psi(y,\varphi)$ only along a line in the complex α plane which is at an angle of $\varphi_\alpha = \pi/2 - \varphi$ to the real axis. That is, complete information on the quantum state, as given by the knowledge of $\Phi(\alpha;s)$ for all values of φ_α in a 2π interval can only be provided by knowing $\Psi(y,\varphi)$ for all values of φ in a π interval. This just expresses the fact that the quantum state contains potential information on measurements of all possible observables, whereas $\Psi(y,\varphi)$ for chosen φ contains that information only for a specific choice of the observable $\hat{x}(\varphi)$.

The correspondence between the phase-space function $P(\alpha;s)$ and the phase-rotated quadrature distributions $p(x,\varphi)$ is analogous. Whereas $p(x,\varphi)$ can be determined from $P(\alpha;s)$, the determination of $P(\alpha;s)$ requires knowledge of $p(x,\varphi)$ for all values of φ within a π interval. The first statement is again not surprising, because $p(x,\varphi)$ is the expectation value of $|x,\varphi\rangle\langle x,\varphi|$ and any expectation value can be calculated by means of a phase-space function $P(\alpha;s)$. Combining Eqs (7.6) and (7.12) yields

$$p(x,\varphi) = \frac{1}{2\pi}\int \mathrm{d}y\, \exp\big(-iyx - \tfrac{1}{2}sy^2\big)\Phi\big(iye^{-i\varphi};s\big). \tag{7.15}$$

Using Eq. (4.92) in Eq. (7.15) to express $\Phi(\alpha;s)$ in terms of $P(\alpha;s)$, we obtain

$$p(x,\varphi) = \frac{1}{2\pi}\int \mathrm{d}y\, \exp\big(-iyx - \tfrac{1}{2}sy^2\big)\int \mathrm{d}^2\alpha\, \exp[iy\langle\alpha|\hat{x}(\varphi)|\alpha\rangle]\, P(\alpha;s), \tag{7.16}$$

where the relation

$$\langle\alpha|\hat{x}(\varphi)|\alpha\rangle = \alpha e^{i\varphi} + \alpha^* e^{-i\varphi} \tag{7.17}$$

[cf. Eq. (3.203)] has been used. When we assume that $s \geq 0$ (or Re $s \geq 0$), then in Eq. (7.16) the integration over y can be performed to obtain

$$p(x,\varphi) = \int \mathrm{d}^2\alpha\, p[x - \langle\alpha|\hat{x}(\varphi)|\alpha\rangle; s] P(\alpha; s), \tag{7.18}$$

where the function $p(x;s)$ is defined by

$$p(x;s) = \frac{1}{2\pi}\int \mathrm{d}y\, \exp\left(-iyx - \tfrac{1}{2}sy^2\right) = \frac{1}{\sqrt{2\pi s}}\exp\left(-\frac{x^2}{2s}\right), \tag{7.19}$$

which in the limit as $s \to 0$ reduces to a δ function:

$$\lim_{s\to 0} p(x;s) = \delta(x). \tag{7.20}$$

In Eq. (7.18) the Gaussian $p[x - \langle\alpha|\hat{x}(\varphi)|\alpha\rangle; s]$ is obviously the associated c-number function of the projector $|x,\varphi\rangle\langle x,\varphi|$ put in s order ($s \geq 0$).[4] In particular, when $s = 1$ (Glauber–Sudarshan representation) the function $p[x - \langle\alpha|\hat{x}(\varphi)|\alpha\rangle; 1]$ is just the phase-rotated quadrature distribution of a coherent state $|\alpha\rangle$ [cf. Eq. (3.202)].

Let us now turn to the problem of determining $P(\alpha;s)$ from $p(x,\varphi)$. Substituting in Eq. (4.93) for $\Phi(\beta;s)$ the result of Eq. (7.14) and using Eq. (7.7), we derive, on changing the integration variables and applying the symmetry relation (7.8),

$$P(\alpha;s) = \frac{1}{\pi^2}\int_\pi \mathrm{d}\varphi \int \mathrm{d}y\, |y| e^{\frac{1}{2}sr^2} \exp[-iy\langle\alpha|\hat{x}(\varphi)|\alpha\rangle] \int \mathrm{d}x\, e^{iyx} p(x,\varphi). \tag{7.21}$$

From inspection of Eq. (7.21) we see that when $s < 0$ (or Re $s < 0$), then the y integration can be performed separately to obtain

$$P(\alpha;s) = \int_\pi \mathrm{d}\varphi \int \mathrm{d}x\, \tilde{p}[x - \langle\alpha|\hat{x}(\varphi)|\alpha\rangle; s] p(x,\varphi), \tag{7.22}$$

where we have introduced the function

$$\tilde{p}(x;s) = \frac{2}{\pi^2}\int_0^\infty \mathrm{d}r\, r e^{\frac{1}{2}sr^2}\cos(rx) = \frac{2}{\pi^2|s|}\,\mathrm{F}\left(1, \frac{1}{2}; -\frac{x^2}{2|s|}\right) \tag{7.23}$$

[$\mathrm{F}(a,b;x)$, confluent hyper-geometric function]. Note that the integral of this function vanishes:

$$\int \mathrm{d}x\, \tilde{p}(x;s) = 0. \tag{7.24}$$

4) Note that if one uses Eq. (7.18) together with (7.19) for values of s with $s < 0$, then one has to deal with a highly singular function $p(x;s)$, which reflects the fact that the associated c-number function of the projector $|x,\varphi\rangle\langle x,\varphi|$ put in s order with $s < 0$ is not well behaved. In practice this difficulty can be overcome by preserving the original order of integration.

For $s>0$ the application of Eq. (7.22) together with Eq. (7.23) would require dealing with a highly singular function $\tilde{p}(x;s)$. We recall that, for positive values of s, depending on the analytic form of $p(x,\varphi)$, the distribution $P(\alpha;s)$ may become highly singular as well, whereas for negative values of s it is well behaved in any case.

Applying Eq. (4.79) to the density operator $\hat{\varrho}$ and and making use of Eqs (7.11) and (7.14), we arrive, on changing the integration variables and applying the symmetry relation (7.9), at the following expansion of $\hat{\varrho}$

$$\begin{aligned}\hat{\varrho} &= \frac{1}{\pi}\int_{\pi}\mathrm{d}\varphi\int\mathrm{d}y\,|y|e^{\frac{1}{2}sr^2}\Psi(-y,\varphi)\hat{D}\big(iye^{-i\varphi};-s\big)\\ &= \frac{1}{\pi}\int_{\pi}\mathrm{d}\varphi\int\mathrm{d}y\,|y|\Psi(-y,\varphi)\exp[iy\hat{x}(\varphi)],\end{aligned}\tag{7.25}$$

which can be represented in the equivalent form as

$$\hat{\varrho} = \frac{2}{\pi}\int_{\pi}\mathrm{d}\varphi\int_0^{\infty}\mathrm{d}y\,|y|\,\{\langle\cos[y\hat{x}(\varphi)]\,\rangle\cos[y\hat{x}(\varphi)] + \langle\sin[y\hat{x}(\varphi)]\,\rangle\sin[y\hat{x}(\varphi)]\}\,.\tag{7.26}$$

A comparison with Eq. (7.1) shows that the operators $\cos[y\hat{x}(\varphi)]$ and $\sin[y\hat{x}(\varphi)]$, for all positive values of y and all values of φ within a π interval, play the role of the operators $\hat{A}_i$ in Eq. (7.1). Expressing in Eq. (7.25) $\Psi(-y,\varphi)$ in terms of $p(x,\varphi)$ according to Eq. (7.7), we may (formally) relate $\hat{\varrho}$ to $p(x,\varphi)$ as

$$\hat{\varrho} = \int_{\pi}\mathrm{d}\varphi\int\mathrm{d}x\,\hat{K}(x,\varphi)p(x,\varphi),\tag{7.27}$$

where the kernel operator is given by

$$\hat{K}(x,\varphi) = \frac{1}{\pi}\int\mathrm{d}y\,|y|\exp\{iy[\hat{x}(\varphi)-x]\}\tag{7.28}$$

[note that $\hat{K}(-x,\varphi\pm\pi)=\hat{K}(x,\varphi)$]. A comparison with Eq. (7.1) shows that now the projectors $|x,\varphi\rangle\langle x,\varphi|$ for all values of x and all values of φ within a π interval play the role of the operators $\hat{A}_i$ in Eq. (7.1).

The extension of Eqs (7.25) and (7.27) to multi-mode fields is straightforward. The single-mode phase-rotated quadrature distributions are replaced by multi-mode joint probability distributions and the kernel operators are replaced by the corresponding multi-mode kernel operators, which are the direct products of the single-mode kernel operators. In particular, for a two-mode system Eq. (7.25) extends to

$$\begin{aligned}\hat{\varrho} = \frac{1}{\pi^2}\int_{\pi}\mathrm{d}\varphi_1\int_{\pi}\mathrm{d}\varphi_2\int\mathrm{d}y_1\int\mathrm{d}y_2\,\{|y_1||y_2|\\ \times\,\Psi(-y_1,-y_2,\varphi_1,\varphi_2)\exp[iy_1\hat{x}_1(\varphi_1)+iy_2\hat{x}_2(\varphi_2)]\},\end{aligned}\tag{7.29}$$

and Eqs (7.27) and (7.28) extend to

$$\hat{\varrho} = \int_\pi \mathrm{d}\varphi_1 \int_\pi \mathrm{d}\varphi_2 \int \mathrm{d}x_1 \int \mathrm{d}x_2\, \hat{K}(x_1, x_2, \varphi_1, \varphi_2) p(x_1, x_2, \varphi_1, \varphi_2), \qquad (7.30)$$

$$\hat{K}(x_1, x_2, \varphi_1, \varphi_2) = \frac{1}{\pi^2} \int \mathrm{d}y_1 \int \mathrm{d}y_2\, |y_1||y_2| \exp\{iy_1[\hat{x}_1(\varphi_1) - x_1] + iy_2[\hat{x}_2(\varphi_2) - x_2]\}. \qquad (7.31)$$

Instead, the probability distributions $p(x, \vartheta, \varphi_1, \varphi_2)$ of the weighted sums

$$\hat{x}(\vartheta, \varphi_1, \varphi_2) = \hat{x}_1(\varphi_1) \cos\vartheta + \hat{x}_2(\varphi_2) \sin\vartheta \qquad (7.32)$$

can be considered, with ϑ being within a $\pi/2$ interval. The probability distributions of the sums are related to the joint probability distributions as

$$p(x, \vartheta, \varphi_1, \varphi_2) = \int \mathrm{d}x_1 \int \mathrm{d}x_2\, p(x_1, x_2, \varphi_1, \varphi_2) \delta(x - x_1 \cos\vartheta - x_2 \sin\vartheta), \qquad (7.33)$$

from which it follows that their characteristic functions (Fourier transforms) $\Psi(y_1, y_2, \varphi_1, \varphi_2)$ and $\Psi(y, \vartheta, \varphi_1, \varphi_2)$ are related to each other as

$$\Psi(y_1 = y\cos\vartheta, y_2 = y\sin\vartheta, \varphi_1, \varphi_2) = \Psi(y, \vartheta, \varphi_1, \varphi_2). \qquad (7.34)$$

7.1.2 Wigner function

Application of Eq. (7.18) to the case where $s=0$ yields the phase-rotated quadrature distributions expressed in terms of the Wigner function. With the help of Eq. (7.20) it is not difficult to prove, on changing the integration variables, that

$$p(x, \varphi) = \tfrac{1}{2} \int \mathrm{d}y\, W\big[\tfrac{1}{2}x\cos\varphi + y\sin\varphi + i\big(-\tfrac{1}{2}x\sin\varphi + y\cos\varphi\big)\big]. \qquad (7.35)$$

Equation (7.35) reveals that the (scaled) phase-rotated quadrature distributions can be regarded as marginals of the Wigner function. An integral relation of the form given in Eq. (7.35) is also called Radon transformation. Inverse Radon transformation then yields the Wigner function in terms of the phase-rotated quadrature distributions for all phases within a π interval.

Application of Eq. (7.21) [together with Eq. (7.17)] to the case where $s = 0$ yields the inverse Radon transformation in the form of

$$W(\alpha) = \frac{1}{\pi^2} \int_\pi \mathrm{d}\varphi \int \mathrm{d}y \int \mathrm{d}x\, |y| \exp[iy(x - 2\alpha_\mathrm{R}\cos\varphi + 2\alpha_\mathrm{I}\sin\varphi)] p(x, \varphi). \qquad (7.36)$$

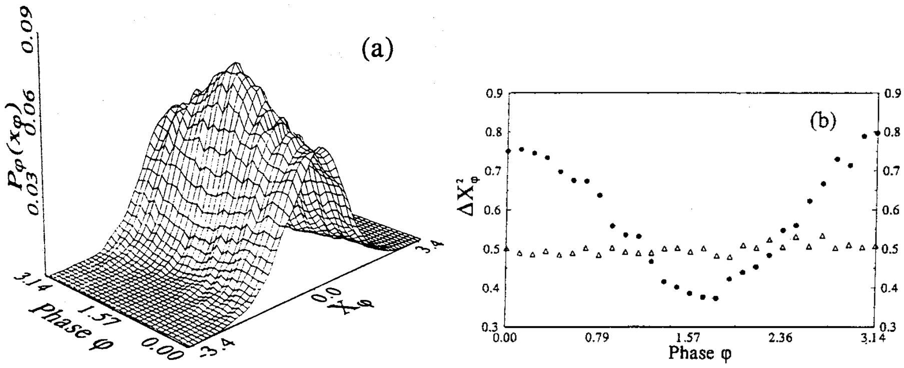

Fig. 7.1 (a) In balanced four-port homodyning measured phase-rotated quadrature distributions $[X_\varphi = x/\sqrt{2},\ P_\varphi(X_\varphi) = \sqrt{2}\, p(x = \sqrt{2} X_\varphi)]$ of a squeezed state. (b) Measured variances $\Delta X_\varphi^2 = \langle [\Delta \hat{x}(\varphi)]^2 \rangle / 2$ of the phase-rotated quadratures: circles, squeezed state; triangle, vacuum state. In the experiment 4000 repeated measurements of the photoelectron difference number at 27 values of the relative phase φ were made. [After Smithey, Beck, Raymer and Faridani (1993).]

Performing the y integral first would lead to an integral kernel which is not well-behaved. In the numerical calculation, regularization techniques can of course be used in order to overcome this difficulty. In the filtered back projection algorithm, the y integral is truncated such that

$$W(\alpha) \simeq W_{\mathrm{C}} = \int_\pi \mathrm{d}\varphi \int \mathrm{d}x\, K_{\mathrm{C}}(x - 2\alpha_{\mathrm{R}} \cos\varphi + 2\alpha_{\mathrm{I}} \sin\varphi) p(x, \varphi), \tag{7.37}$$

where

$$K_{\mathrm{C}}(x) = \frac{1}{\pi^2} \int_{-y_{\mathrm{C}}}^{y_{\mathrm{C}}} \mathrm{d}y\, |y| e^{iyx} \tag{7.38}$$

($y_{\mathrm{C}} > 0$). In the first experimental demonstration of the method [Smithey, Beck, Raymer and Faridani (1993)], a pulsed signal field was superimposed by a pulsed local-oscillator field much stronger than the signal field, and the measurements and reconstructions were performed for a squeezed signal field (Figs 7.1 and 7.2) and for a vacuum signal field, the field mode detected being selected by the spatio-temporal mode of the local-oscillator field.

As mentioned in Section 6.5.4, the measured phase-rotated quadrature distributions do not correspond, in general, to the true signal mode, but they must be regarded as the distributions of a superposition of the signal and an additional noise source [Eq. (6.272)], because of nonperfect detection. Substituting in Eq. (7.35) for $p(x, \varphi)$ the measured distributions $p(x, \varphi; \eta)$ with $\eta < 1$ [Eq. (6.271)] and performing the inverse Radon transform on them, the Wigner function of a noise-assisted signal field is effectively reconstructed. Equivalently, the reconstructed Wigner function can be regarded

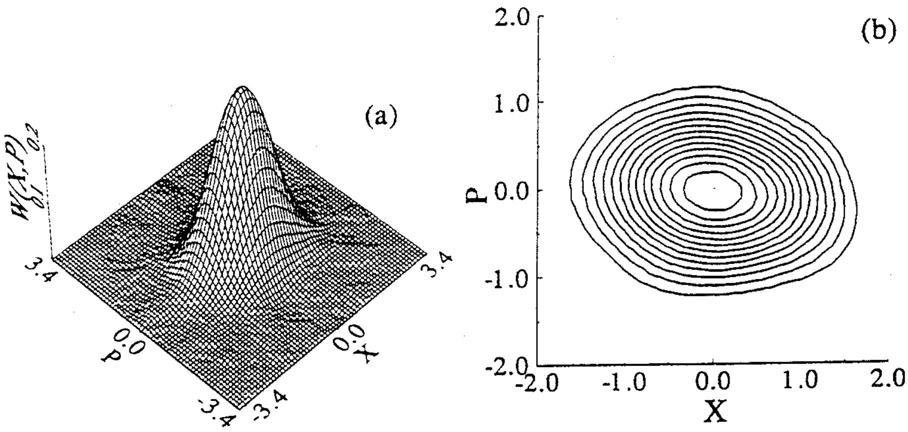

Fig. 7.2 Wigner distributions $W(X,P)=2^{-1}W[\alpha=2^{-1/2}(X+iP)]$ reconstructed from the measured rotated-phase quadrature distributions for a squeezed state, viewed in (a) 3D and as (b) contour plots, with equal numbers of constant-height contours. The reconstruction is performed by using inverse Radon transformation. [After Smithey, Beck, Raymer and Faridani (1993).]

as an s-parameterized phase-space function of the true signal field, however with $s<0$.

From Eq. (6.271) together with Eq. (6.268) it follows that the characteristic function $\Psi(y,\varphi;\eta)$ of $p(x,\varphi;\eta)$ typically measured when $\eta<1$ [Eq. (7.7) with $p(x,\varphi;\eta)$ in place of $p(x,\varphi)$] is related to the characteristic function $\Psi(y,\varphi)$ of $p(x,\varphi)$ as

$$\Psi(y,\varphi;\eta) = \exp\left[\tfrac{1}{2}(1-\eta^{-1})y^2\right]\Psi(y,\varphi). \tag{7.39}$$

Combining Eqs (7.12) and (7.39), we see that $\Psi(y,\varphi;\eta)$ and $\Phi(\beta;s)$ are related to each other as

$$\Psi(y,\varphi;\eta) = \exp\left[-\tfrac{1}{2}(s-1+\eta^{-1})y^2\right]\Phi\left(iye^{-i\varphi};s\right). \tag{7.40}$$

Hence, in the exponentials in Eqs (7.16) and (7.21) for s performing the substitution $s-1+\eta^{-1}$ yields the relations between $P(\alpha;s)$ and $p(x,\varphi;\eta)$. From Eq. (7.21) it then follows that, on recalling Eq. (7.17),

$$P(\alpha;s) = \frac{1}{\pi^2}\int_\pi \mathrm{d}\varphi \int \mathrm{d}y \int \mathrm{d}x \, \Big\{\exp\left[\tfrac{1}{2}(s-1+\eta^{-1})y^2\right] \\ \times\ |y|\exp[iy(x-\alpha_\mathrm{R}\cos\varphi+\alpha_\mathrm{I}\sin\varphi)]p(x,\varphi;\eta)\Big\}. \tag{7.41}$$

Obviously, when $s = 1 - \eta^{-1}$, then Eq. (7.41) takes the form of Eq. (7.36),

$$P(\alpha; s = 1 - \eta^{-1}) \\ = \frac{1}{\pi^2} \int_\pi \mathrm{d}\varphi \int \mathrm{d}y \int \mathrm{d}x \, |y| \exp[iy(x - 2\alpha_\mathrm{R} \cos\varphi + 2\alpha_\mathrm{I} \sin\varphi)] p(x, \varphi; \eta). \tag{7.42}$$

Accordingly, Eq. (7.35) takes the form

$$p(x, \varphi; \eta) = \\ \tfrac{1}{2} \int \mathrm{d}y \, P\left[\tfrac{1}{2} x \cos\varphi + y \sin\varphi + i\left(-\tfrac{1}{2} x \sin\varphi + y \cos\varphi\right); s = 1 - \eta^{-1}\right]. \tag{7.43}$$

Thus, in Eq. (7.35) replacing $p(x, \varphi)$ with $p(x, \varphi; \eta)$, the inverse Radon transformation yields the signal-mode phase-space function $P(\alpha; s = 1 - \eta^{-1})$ in place of the Wigner function.

7.2
Density matrix in phase-rotated quadrature basis

Let us reconstruct the density-matrix elements in a phase-rotated quadrature basis of chosen phase φ, $\langle x, \varphi | \hat{\varrho} | x', \varphi \rangle$, from the phase-rotated quadrature distributions $p(\tilde{x}, \tilde{\varphi})$ for all phases $\tilde{\varphi}$ in a π interval. The most straightforward way is to extend the expression (7.5) for $|x\rangle\langle x|$ to $|x'\rangle\langle x|$. Recalling the commutation relation (3.145), we may write, according to the position representation of the momentum operator in quantum mechanics,

$$\frac{\partial}{\partial x} |x, \varphi\rangle = -\frac{i}{2} \, \hat{x}\left(\varphi - \tfrac{1}{2}\pi\right) |x, \varphi\rangle, \tag{7.44}$$

from which it follows that

$$|x, \varphi\rangle = \exp\left[-\frac{i}{2} (x - x') \, \hat{x}\left(\varphi - \tfrac{1}{2}\pi\right)\right] |x', \varphi\rangle \\ = \hat{D}\left[\tfrac{1}{2}(x - x') e^{-i\varphi}\right] |x', \varphi\rangle. \tag{7.45}$$

Hence we may write

$$|x', \varphi\rangle\langle x, \varphi| = \hat{D}\left[\tfrac{1}{2}(x' - x) e^{-i\varphi}\right] |x, \varphi\rangle\langle x, \varphi|. \tag{7.46}$$

Combining Eqs (7.5) and (7.46) and applying the relation (3.53), it is not difficult to prove that

$$|x', \varphi\rangle\langle x, \varphi| = \frac{1}{2\pi} \int \mathrm{d}y \, e^{-iy(x+x')/2} \hat{D}\left[\left(\tfrac{1}{2}(x' - x) + iy\right) e^{-i\varphi}\right], \tag{7.47}$$

so that the density matrix in a phase-rotated quadrature basis can be given by ($x = x_1 - x_2$, $x' = x_1 + x_2$)

$$\langle x_1 - x_2, \varphi | \hat{\varrho} | x_1 + x_2, \varphi \rangle = \frac{1}{2\pi} \int \mathrm{d}y \, e^{-ix_1 y} \Psi(y, x_2, \varphi), \tag{7.48}$$

where the characteristic function $\Psi(y, x_2, \varphi)$ reads as

$$\Psi(y, x_2, \varphi) = \mathrm{Tr}\big\{\hat{\varrho} \hat{D}\big[(x_2 + iy)e^{-i\varphi}\big]\big\}. \tag{7.49}$$

Recalling Eq. (7.10), it can be related to the characteristic function of a phase-rotated quadrature distribution,

$$\Psi(y, x_2, \varphi) = \Psi(\tilde{y}, \tilde{\varphi}), \tag{7.50}$$

with

$$\tilde{y} = \sqrt{y^2 + x_2{}^2}\,, \qquad \tilde{\varphi} = \varphi - \arg(y - ix_2). \tag{7.51}$$

Thus, combining Eqs (7.48) and (7.50) and making use of Eq. (7.7), we obtain the density-matrix elements from the phase-rotated quadrature distributions by means of a two-fold Fourier transformation [Kühn, Welsch and Vogel (1994)]:

$$\langle x_1 - x_2, \varphi | \hat{\varrho} | x_1 + x_2, \varphi \rangle = \frac{1}{2\pi} \int \mathrm{d}y \, e^{-ix_1 y} \int \mathrm{d}x \, e^{i\tilde{y}x} p(x, \tilde{\varphi}). \tag{7.52}$$

Note that Eq. (7.52) can be used to obtain the density matrix in different representations, by varying the phase φ of the quadrature component defining the basis. Furthermore, Eq. (7.52) can also be extended, in principle, to imperfect detection, expressing $\Psi(\tilde{y}, \tilde{\varphi})$ in terms of $\Psi(\tilde{y}, \tilde{\varphi}; \eta)$ according to Eq. (7.39). As an illustration of the method, in Fig. 7.3 the reconstructed density matrices in (a) the "position" basis, $\varphi = 0$, and (b) the "momentum" basis, $\varphi = \pi/2$, of a squeezed vacuum state are shown.[5] The phases $\varphi = 0$ and $\varphi = \pi/2$ coincide with the phases of minimal and maximal field noise respectively.

Using Eq. (7.12), we rewrite Eq. (7.50) as

$$\Psi(y, x_2, \varphi) = e^{-\frac{1}{2}s\tilde{y}^2} \Phi\big(i\tilde{y}e^{-i\tilde{\varphi}}; s\big) = e^{-\frac{1}{2}s\tilde{y}^2} \Phi\big[(x_2 + iy)e^{-i\varphi}; s\big]. \tag{7.53}$$

Substituting this expression into Eq. (7.48) and using Eq. (4.92), we may express the density matrix in a phase-rotated quadrature basis in terms of an

5) The homodyne data were recorded by T. Coudreau, A.Z. Khoury and E. Giacobino, using the experimental setup reported by Lambrecht, Coudreau, Steinberg and Giacobino (1996). In the experiment the quadratures were measured at 48 phases and at each phase 7812 measurements were performed. The reconstruction is based on the numerical algorithm by Zucchetti, Vogel, Tasche and Welsch (1996).

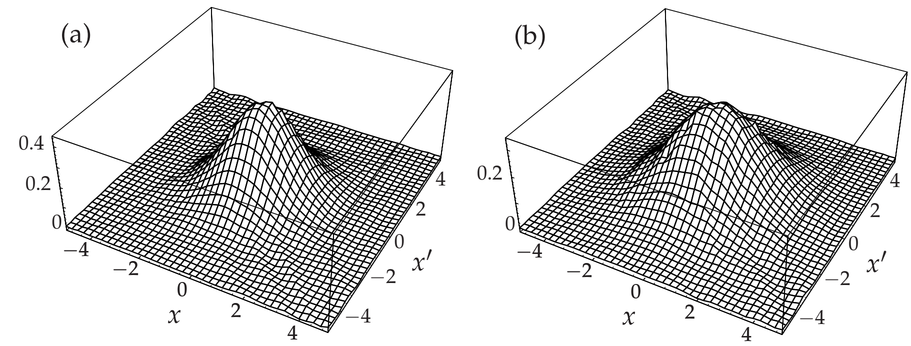

Fig. 7.3 From measured phase-rotated quadrature distributions, reconstructed (real) density matrix $\langle x,\varphi|\hat{\varrho}|x',\varphi\rangle$ of a squeezed state in (a) the "position" basis ($\varphi=0$) and (b) the "momentum" basis ($\varphi=\pi/2$).

s-parameterized phase-space function:

$$\langle x_1 - x_2, \varphi|\hat{\varrho}|x_1 + x_2, \varphi\rangle = \frac{1}{2\pi}\int \mathrm{d}y\, e^{-ix_1y-\frac{1}{2}s\tilde{y}^2} \int \mathrm{d}^2\alpha\, \exp\left[(x_2+iy)e^{-i\varphi}\alpha^* - \text{c.c.}\right] P(\alpha; s), \tag{7.54}$$

which, on changing integration variables, can be rewritten as

$$\langle x_1 - x_2, \varphi|\hat{\varrho}|x_1 + x_2, \varphi\rangle = \frac{1}{2\pi}\int \mathrm{d}z\, \Big\{e^{-ix_1z-\frac{1}{2}s\tilde{z}^2} \times \int \mathrm{d}x \int \mathrm{d}y\, e^{2i(xz-x_2y)} P[x\cos\varphi + y\sin\varphi + i(-x\sin\varphi + y\cos\varphi); s]\Big\}, \tag{7.55}$$

with $\tilde{y}=\tilde{y}(y,x_2)$ and $\tilde{z}=\tilde{z}(z,x_2)$ being defined according to Eq. (7.51). If $s>0$ (or $\mathrm{Re}\, s>0$), then in Eq. (7.55) the integration over z can be performed separately to reduce the three-fold integral transformation to a two-fold one. In the limit as $s\to 0$ a δ function appears and thus the integration over x can also be performed. In this way we find that the density-matrix elements can be obtained from the Wigner function by a single Fourier transformation as follows [Wigner (1932)]:

$$\langle x_1 - x_2, \varphi|\hat{\varrho}|x_1 + x_2, \varphi\rangle = \tfrac{1}{2}\int \mathrm{d}y\, e^{-2ix_2y} W\left[\tfrac{1}{2}x_1\cos\varphi + y\sin\varphi + i\left(-\tfrac{1}{2}x_1\sin\varphi + y\cos\varphi\right)\right], \tag{7.56}$$

which, when $x_2=0$, just reduces to Eq. (7.35).

7.3 Density matrix in the number basis

If a phase-space function $P(\alpha;s)$ is known, then the density-matrix elements in the number basis, $\langle m|\hat{\varrho}|n\rangle$, can be calculated, in principle, by applying Eq. (4.85) and calculating the c-number function $\pi\langle m|\hat{\delta}(\hat{a}-\alpha;-s)|n\rangle$ associated with the operator $|n\rangle\langle m|$ in s order. From Section 6.5.5 we know that $P(\alpha;s)$ can be measured directly for $s\leq -1$. Thus, the procedure outlined could be used to infer $\langle m|\hat{\varrho}|n\rangle$ from the measured $P(\alpha;s)$. Since the resulting reconstruction formula is not suited for statistical sampling [according to Eq. (7.2)] and the inaccuracies of the measured $P(\alpha;s)$ can give rise to an error explosion in the reconstructed density-matrix elements, the method has been of less experimental relevance. Instead of using integration techniques, one could try to apply equivalent differentiation techniques. In particular, combining Eq. (4.67) with Eqs (3.22) and (3.59), we easily derive

$$Q(\alpha) = \pi^{-1} e^{-|\alpha|^2} \sum_{m,n} \frac{1}{\sqrt{m!n!}} \alpha^{*m} \alpha^{n} \langle m|\hat{\varrho}|n\rangle, \tag{7.57}$$

from which it follows that

$$\langle m|\hat{\varrho}|n\rangle = \frac{\pi}{\sqrt{m!n!}} \frac{\partial^{m+n}}{\partial\alpha^{*m}\partial\alpha^{n}} e^{|\alpha|^2} Q(\alpha)\Big|_{\alpha=\alpha^*=0}. \tag{7.58}$$

Clearly, the inaccuracies in the measured Q function prevent one from carrying out the derivatives with sufficient precision, in general. Therefore, much effort has been made to obtain the density matrix in the number basis from measurable data with sufficient precision and as directly as possible.

7.3.1 Sampling from quadrature components

It turns out that the density-matrix elements in the number basis can be inferred from the measured phase-rotated quadrature distributions in a very direct way. In the number basis Eq. (7.27) reads as

$$\varrho_{mn} = \langle m|\hat{\varrho}|n\rangle = \int_{\pi} \mathrm{d}\varphi \int \mathrm{d}x\, K_{mn}(x,\varphi)\, p(x,\varphi), \tag{7.59}$$

where the kernel function (also called the sampling function) can be written as

$$K_{mn}(x,\varphi) = \frac{1}{\pi} \int \mathrm{d}y\, |y| \langle m| \exp\{iy[\hat{x}(\varphi)-x]\}|n\rangle. \tag{7.60}$$

Recalling the relation (3.195) [together with Eq. (3.194)], we find that $K_{mn}(x,\varphi)$ takes the form of

$$K_{mn}(x,\varphi) = e^{-i(m-n)\varphi} f_{mn}(x), \tag{7.61}$$

where

$$f_{mn}(x) = \frac{1}{\pi} \int \mathrm{d}y\, |y| \langle m| \exp[iy(\hat{x} - x)] |n\rangle \tag{7.62}$$

$[\hat{x} \equiv \hat{x}(0)]$. Recalling the relation $\hat{K}(x, \varphi + \pi) = \hat{K}(-x, \varphi)$, we see that

$$f_{mn}(-x) = (-1)^{m-n} f_{mn}(x). \tag{7.63}$$

In order to find an expression of $f_{mn}(x)$, which is more suitable for practical applications than that in Eq. (7.62), let us express the phase-rotated quadrature distributions in terms of the density-matrix elements in the number basis,

$$p(x, \varphi) = \sum_{k,l} \langle x, \varphi | k \rangle \langle l | x, \varphi \rangle \varrho_{kl}. \tag{7.64}$$

Recalling Eqs (3.193) and (3.199), we may rewrite Eq. (7.64) as

$$p(x, \varphi) = \sum_{k,l} g_{kl}(x) e^{i(k-l)\varphi} \varrho_{kl}, \tag{7.65}$$

where

$$g_{kl}(x) = \psi_k(x) \psi_l(x) \tag{7.66}$$

$[\psi_k(x) = \langle k|x\rangle]$. We now multiply Eq. (7.65) by $f_{mn}(x) e^{-i(m-n)\varphi}$ and integrate with respect to φ and x. Using the symmetry relation (7.63), we can easily derive

$$\begin{aligned} &\int_{\pi} \mathrm{d}\varphi \int \mathrm{d}x\, p(x, \varphi) f_{mn}(x) e^{-i(m-n)\varphi} \\ &\qquad = \tfrac{1}{2} \int_{2\pi} \mathrm{d}\varphi \int \mathrm{d}x\, p(x, \varphi) f_{mn}(x) e^{-i(m-n)\varphi} \\ &\qquad = \sum_{k,l} \Big[\pi \int \mathrm{d}x\, g_{kl}(x) f_{mn}(x)\Big] \varrho_{kl}\,, \qquad k - l = m - n. \end{aligned} \tag{7.67}$$

Comparing with Eq. (7.59) [together with Eq. (7.61)], we find that $f_{mn}(x)$ obeys the integral equation

$$\pi \int \mathrm{d}x\, g_{kl}(x) f_{mn}(x) = \delta_{km} \delta_{ln}\,, \qquad k - l = m - n. \tag{7.68}$$

Thus, the sought function $f_{mn}(x)$ is orthonormal to the product of wave functions $\psi_k(x)\psi_l(x)$. Obviously, $\psi_k(x)$ is the ordinary (i.e., regular) solution of the harmonic-oscillator Schrödinger equation for the kth energy level,

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(k + \tfrac{1}{2} - \tfrac{1}{4}x^2\right)\right] \phi_k(x) = 0 \tag{7.69}$$

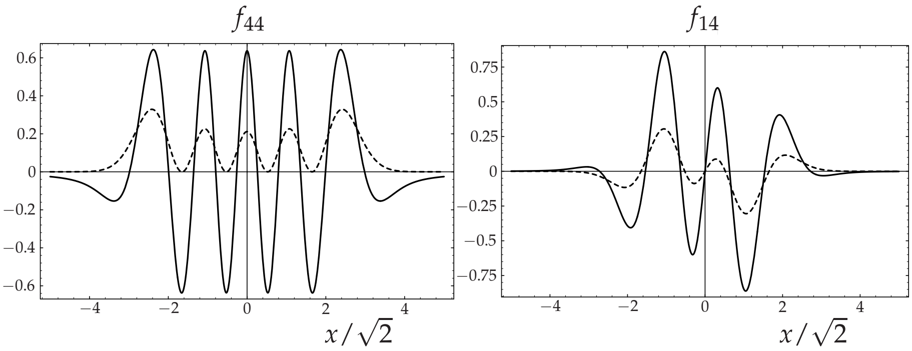

Fig. 7.4 Examples of the kernel function $\pi f_{mn}(x)$ for sampling the density-matrix elements ϱ_{mn} in the number basis from the phase-rotated quadrature distributions according to Eq. (7.59) together with Eq. (7.61) for $m=n=4$ (left figure) and $m=1$, $n=4$ (right figure). For comparison, the product of the wave functions $\psi_m(x)\psi_n(x)/\sqrt{2}$ is shown (dashed lines). [After Leonhardt (1997).]

$[\phi_k(x) = \psi_k(x)]$. It is worth noting that the solution of the integral equation (7.68) can be given in the form of the derivative of a product of regular and irregular wave functions [Leonhardt, Munroe, Kiss, Richter and Raymer (1996); Richter (1996a)],

$$f_{mn}(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\psi_m(x)\chi_n(x)], \tag{7.70}$$

where $\chi_n(x)$ is an irregular solution of the wave equation (7.69) $[\phi_n(x) = \chi_n(x)]$ which must be chosen such that

$$\psi_n(x)\,\frac{\mathrm{d}\chi_n(x)}{\mathrm{d}x} - \frac{\mathrm{d}\psi_n(x)}{\mathrm{d}x}\,\chi_n(x) = \frac{2}{\pi} \tag{7.71}$$

(for a proof, see Appendix D). It should be pointed out that the ambiguities of the kernel function leave enough room to choose the most convenient form [for details, see Leonhardt (1997)]. Examples are shown in Fig. 7.4.

Equation (7.59) reveals that, according to Eq. (7.2), the density-matrix elements in the number basis can be sampled directly from the measured phase-rotated quadrature statistics. Each density-matrix element ϱ_{mn} can be regarded as a statistical average of the bounded kernel function $K_{mn}(x,\varphi)$.[6] In an experiment each outcome x of $\hat{x}(\varphi)$, with $\varphi \in [0,\pi)$, contributes individually to ϱ_{mn}, so that ϱ_{mn} is gradually building up during the data collection. That is to say, ϱ_{mn} can be sampled from a sufficiently large set of homodyne data in real time, and the mean value obtained from different experiments can be expected to be normal-Gaussian distributed around the true value, because of the central-limit theorem. Moreover, the sampling method can also

6) For a numerical implementation of Eq. (7.70), see Leonhardt, Munroe, Kiss, Richter and Raymer (1996).

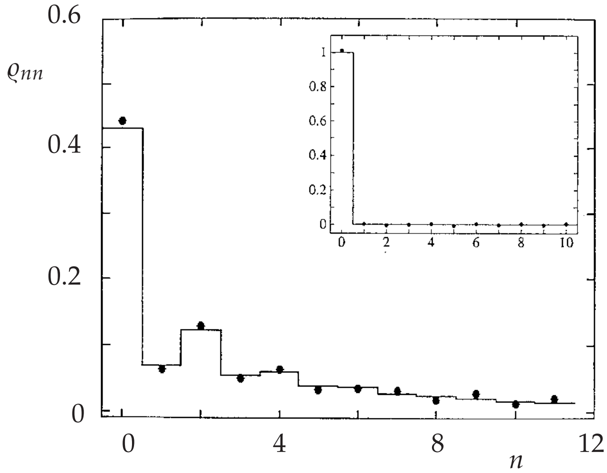

Fig. 7.5 From the phase-rotated quadrature distributions according to Eq. (7.59) reconstructed photon-number distribution of a squeezed vacuum and the vacuum state (inset). Solid points refer to experimental data, histograms to theory. [After Schiller, Breitenbach, Pereira, Müller and Mlynek (1996).]

be used to estimate the statistical error. Experimentally, the method was first successfully applied to the determination of the density matrix of squeezed light generated by a continuous-wave optical parametric amplifier [Schiller, Breitenbach, Pereira, Müller and Mlynek (1996)]. The reconstructed diagonal density-matrix elements are shown in Fig. 7.5.

Since in a realistic experiment the quantum efficiency η is always less than unity, insertion into Eq. (7.59) of the measured phase-rotated quadrature distributions $p(x,\varphi;\eta)$ [in place of the exact distributions $p(x,\varphi)$] yields the density matrix of a noise-assisted state in general (cf. the last paragraph in Section 7.1.2). If $\eta > 0.5$ it is possible to compensate for detection losses by introducing a modified sampling function which depends on η such that performing the sampling algorithm on the real measured (i. e., the smeared) phase-rotated quadrature distributions $p(x,\varphi;\eta)$ yield the correct quantum state [D'Ariano, Leonhardt and Paul (1995)]. Another approach to the problem of loss compensation is that the sampling function is left unchanged and, at the first stage, the density matrix of the quantum state which corresponds to the measured distributions $p(x,\varphi;\eta)$ is reconstructed. After that, at the second stage, the true density matrix is calculated from the reconstructed one using an inverse Bernoulli transformation [Kiss, Herzog and Leonhardt (1995)].

Since the density matrix in any basis contains the full information about the quantum state of the system under consideration, all quantum-statistical properties can be inferred from it. Let $\hat{F}$ be an operator whose expectation value

$$\langle\hat{F}\rangle = \sum_{mn} F_{nm}\varrho_{mn}\,, \tag{7.72}$$

is to be determined. One may be tempted to calculate it from the reconstructed density matrix. However, an experimentally determined density matrix always suffers from various inaccuracies which can propagate (and increase) in the calculation process. Therefore, it may be advantageous to determine directly the quantities of interest from the measured data, without reconstructing the whole quantum state. In particular, in Eq. (7.72) substituting the integral representation (7.59) for ϱ_{mn}, one can try to obtain an integral representation

$$\langle\hat{F}\rangle = \int_{\pi} \mathrm{d}\varphi \int \mathrm{d}x\, K_F(x,\varphi)p(x,\varphi) \tag{7.73}$$

suited to the direct sampling of $\langle\hat{F}\rangle$, from the quadrature component distributions $p(x,\varphi)$. It is worth noting that the kernel function $K_F(x,\varphi)$ is not defined uniquely by the integral relation (7.73), but only up to a function $\Theta(x,\varphi)$ that satisfies the integral equation

$$\int_{\pi} \mathrm{d}\varphi \int \mathrm{d}x\, \Theta(x,\varphi)p(x,\varphi) = 0. \tag{7.74}$$

Hence, if the integral kernel $K_F(x,\varphi)$ which is obtained from Eq. (7.72), together with Eq. (7.59), is unbounded for $|x| \to \infty$ such that the x integral in Eq. (7.73) does not exist for any normalizable state, it cannot be concluded that $\langle\hat{F}\rangle$ cannot be sampled from $p(x,\varphi)$, since a different, bounded kernel may exist. Obviously, the ambiguity mentioned is also true for the kernel function $K_{mn}(x,\varphi)$ in Eq. (7.59). In particular, applying normally-ordered moment expansion (Section 7.5), $K_{mn}(x,\varphi)$ can be represented in the equivalent form of Eq. (7.95), which is not necessarily suitable for statistical sampling.

7.3.2 Reconstruction from displaced number states

From Section 6.5.3 we know that in unbalanced homodyning the photon-number distribution of the transmitted signal mode is, under certain conditions, the displaced photon-number distribution of the signal mode, $p_m(\alpha)$, the displacement parameter $\alpha = |\alpha|e^{i\varphi}$ being controlled by the local-oscillator complex amplitude. Expanding the density operator in the number basis, $p_m(\alpha)$ can be related to the density matrix of the signal mode as

$$p_m(\alpha) = \langle m,\alpha|\hat{\varrho}|m,\alpha\rangle = \sum_{k,n}\langle m,\alpha|k\rangle\langle n|m,\alpha\rangle\varrho_{kn}\,, \tag{7.75}$$

where the expansion coefficients $\langle n|m,\alpha\rangle$ can be taken from Eq. (3.101). Equation (7.75) can always be inverted in order to obtain ϱ_{mn} in terms of $p_k(\alpha)$. Combining Eqs (4.62) and (4.80) [together with Eq. (4.81)], we may write

$$\varrho_{mn} = \sum_{k=0}^{\infty} \int \mathrm{d}^2\alpha\, K_{mn}^{k}(\alpha) p_k(\alpha), \tag{7.76}$$

where

$$K_{mn}^{k}(\alpha) = \frac{2}{(1-s)} \left(\frac{s+1}{s-1}\right)^{k} \langle m|\hat{\delta}(\hat{a}-\alpha;-s)|n\rangle. \tag{7.77}$$

It can be shown that $K_{mn}^{k}(\alpha)$ is bounded for $-1<s\leq 0$, which offers the possibility of direct sampling of ϱ_{mn} from the displaced number probability distribution $p_m(\alpha)$ [Mancini, Man'ko and Tombesi (1997)].

Since $p_m(\alpha)$ as a function of α for chosen m already determines the quantum state, it is clear that when m is allowed to be varying, then – in contrast to Eq. (7.76) – $p_m(\alpha)$ need not be known for all complex values of α in order to reconstruct the density-matrix elements ϱ_{kn} from $p_m(\alpha)$. In particular, it is sufficient to know $p_m(\alpha)$ for all values of m and all phases $\varphi = \arg(\alpha)$, $|\alpha|$ being fixed [Leibfried, Meekhof, King, Monroe, Itano and Wineland (1996), Opatrný and Welsch (1997)]. For chosen $|\alpha|$ we can regard $p_m(\alpha)$ as a function of φ and introduce the Fourier coefficients

$$p_m^{k}(|\alpha|) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi\, e^{ik\varphi} p_m(\alpha) \tag{7.78}$$

($k = 0, 1, 2, \ldots$), which are related to the density-matrix elements whose row and column indices differ by k. Substituting Eqs (7.75) into Eq. (7.78) and using Eq. (3.101), we can easily derive

$$p_m^{k}(|\alpha|) = \sum_{n=0}^{\infty} G_{mn}^{k}(|\alpha|) \varrho_{n+k\,n}\,, \tag{7.79}$$

where

$$G_{mn}^{k}(|\alpha|) = e^{-|\alpha|^2} \frac{m!}{\sqrt{n!(n+k)!}} \mathrm{L}_m^{n-m}(|\alpha|^2) \mathrm{L}_m^{k+n-m}(|\alpha|^2) |\alpha|^{2n+k-2m}. \tag{7.80}$$

Inverting Eq. (7.79) for each value of k yields the sought density-matrix elements. Unfortunately, no analytical solution has been found. However, Eq. (7.79) can be inverted numerically, setting $\varrho_{mn} = 0$ for $m, n > n_{\max}$ and using, e. g., least-squares inversion. The method was first used to reconstruct the density matrix of the center-of-mass motion of a trapped ion [Leibfried, Meekhof, King, Monroe, Itano and Wineland (1996)]. Details on the measurement techniques and an example of a reconstructed density matrix are given in Section 13.5.2 (Fig. 13.11, p. 477).

7.4
Local reconstruction of phase-space functions

The displaced photon-number statistics $p_m(\alpha)$ which are measurable in unbalanced homodyning (Section 6.5.3) can be used for a point-wise reconstruction of s-parameterized phase-space functions $P(\alpha; s)$ [Wallentowitz and Vogel (1996); Banaszek and Wódkiewicz (1996)]. From Eq. (4.53) together with (4.62) it is easily seen that

$$P(\alpha; s) = \frac{2}{\pi(1-s)} \sum_{m=0}^{\infty} \left(\frac{s+1}{s-1}\right)^m p_m(\alpha). \tag{7.81}$$

Hence, in principle, all the phase-space functions $P(\alpha; s)$, with $s < 1$, can be obtained from $p_m(\alpha)$ for each phase-space point α in a very direct way, without integral transformations. However, for unknown statistics $p_m(\alpha)$ it is advantageous, in practice, to require nonexploding coefficients, $|(s+1)/(s-1)| \leq 1$, which are obtained for $s \leq 0$ only. In particular when $s = -1$, then Eq. (7.81) reduces to the well-known result that $Q(\alpha) = \pi^{-1} p_0(\alpha)$, with $p_0(\alpha) = \langle\alpha|\hat{\varrho}|\alpha\rangle$. Further, choosing $s = 0$ in Eq. (7.81), we arrive at the Wigner function,

$$W(\alpha) = \frac{2}{\pi} \sum_{m=0}^{\infty} (-1)^m p_m(\alpha). \tag{7.82}$$

Equation (7.82) reflects the fact that the Wigner function is proportional to the expectation value of the displaced parity operator [cf. Eq. (4.64)].

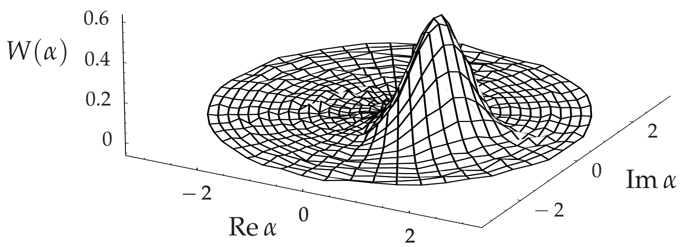

Fig. 7.6 The reconstructed Wigner function $W(\alpha)$ for a weak coherent state with approximately one photon. [After Banaszek, Radzewicz, Wódkiewicz and Krasiński (1999).]

Experimentally, the method was first applied to the reconstruction of the Wigner function of the center-of-mass motion of a trapped ion [Leibfried, Meekhof, King, Monroe, Itano and Wineland (1996)], see Section 13.5.2 (Fig. 13.10). For light, the method was demonstrated experimentally as well [Banaszek, Radzewicz, Wódkiewicz and Krasiński (1999)], an example is given in Fig. 7.6. The use of the method for light is hampered by the problem that it is difficult to discriminate adjacent photon numbers in photodetection needed for determining the Wigner function according to Eq. (7.82). To determine the Wigner function by this method for more general light fields a cascaded homodyne detection scheme has been proposed [Kis, Kiss, Janszky, Adam, Wallentowitz and Vogel (1999)], which combines the unbalanced scheme with balanced homodyning, allowing the determination of the desired photon-number statistics of the displaced field.

7.5 Normally ordered moments

It is often sufficient to know some moments of the creation and annihilation operators rather than the overall quantum state. By means of Eq. (4.96) s-ordered moments can be expressed in terms of normally ordered moments and vice versa, so that we may restrict our attention to normally ordered moments. We apply Eqs (3.19) and (3.20) and express the normally ordered moments in terms of the density-matrix elements in the number basis:

$$\begin{aligned}\langle \hat{a}^{\dagger m}\hat{a}^{n}\rangle &= \sum_{k=0}^{\infty}\langle k|\hat{\varrho}\hat{a}^{\dagger m}\hat{a}^{n}|k\rangle \\ &= \sum_{k=n}^{\infty}\sqrt{k(k-1)\cdots(k-n+1)}\sqrt{(k+1)(k+2)\cdots(k+m-n)}\,\varrho_{k\,k+m-n} \\ &= \sum_{k=n}^{\infty}\sqrt{\frac{(k+m-n)!}{(k-n)!}}\,\varrho_{k\,k+m-n}\,. \end{aligned} \tag{7.83}$$

In Eq. (7.83) substituting for $\varrho_{k\,k+m-n}$ the expression in Eq. (7.59), we expect that $\langle \hat{a}^{\dagger m}\hat{a}^{n}\rangle$ can be expressed in terms of $p(x,\varphi)$ as

$$\langle \hat{a}^{\dagger m}\hat{a}^{n}\rangle = \int_{\pi}\mathrm{d}\varphi\int\mathrm{d}x\,M_{mn}(x,\varphi)p(x,\varphi). \tag{7.84}$$

Indeed, it can be shown that a kernel function $M_{mn}(x,\varphi)$ which solves Eq. (7.84) together with Eq. (7.83) reads [Richter (1996b)]

$$M_{mn}(x,\varphi) = M_{mn}(x)e^{i(m-n)\varphi}, \tag{7.85}$$

where

$$M_{mn}(x) = \frac{m!n!}{\pi\sqrt{2^{m+n}}(m+n)!}\,\mathrm{H}_{m+n}\big(x/\sqrt{2}\big). \tag{7.86}$$

Using Eq. (7.64) together with Eq. (3.199), we may write

$$\begin{aligned}
&\int_\pi \mathrm{d}\varphi \int \mathrm{d}x\, e^{i(m-n)\varphi} \mathrm{H}_{m+n}(x/\sqrt{2})\, p(x,\varphi) \\
&= \sum_{k,l} (\pi 2^{k+l} k! l!)^{-1/2} \varrho_{kl} I_{m+n\,kl} \tfrac{1}{2} \int_{2\pi} \mathrm{d}\varphi\, e^{i(m-n-l+k)\varphi} \\
&= \sum_{k,l} (\pi 2^{k+l} k! l!)^{-1/2} \varrho_{kl} I_{m+n\,kl} \pi \delta_{m-n\,l-k}\,,
\end{aligned} \tag{7.87}$$

where

$$\begin{aligned}
I_{klm} &= \int \mathrm{d}x\, e^{-x^2} \mathrm{H}_k(x) \mathrm{H}_l(x) \mathrm{H}_m(x) \\
&= \frac{\sqrt{\pi}\, 2^n k! l! m!}{(n-k)!(n-l)!(n-m)!}\, \delta_{k+l+m\,2n}\,,
\end{aligned} \tag{7.88}$$

with n being a (non-negative) integer. Starting from the first line in Eq. (7.87) we have used the symmetry relation (7.8) and the property of the Hermite polynomials that $\mathrm{H}_n(-x) = (-1)^n \mathrm{H}_n(x)$. Substitution of the expression (7.88) into Eq. (7.87) and comparison of the result with Eq. (7.83) then shows the validity of Eq. (7.84) together with Eqs (7.85) and (7.86). Equation (7.84) offers the possibility of direct sampling of normally ordered moments from the phase-rotated quadrature distributions.[7]

The extension of the method to the reconstruction of normally-ordered moments of multi-mode fields from the corresponding joint phase-rotated quadrature distributions is straightforward. Instead, they can also be inferred from combined distributions, following the procedure outlined in the last paragraph of Section 7.1.1. The method was first used to experimentally demonstrate the determination of the ultrafast two-time photon number correlation of a nanosecond optical pulse [McAlister and Raymer (1997)]. In the experiment, two femtosecond local-oscillator pulses were used and phase-rotated sum quadratures $\hat{x}(\vartheta, \varphi_1, \varphi_2)$ [Eq. (7.32)] were measured. From these the normalized second-order coherence function

$$g^{(2)}(t_1, t_2) = \frac{\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_2 \rangle}{\langle \hat{a}_1^\dagger \hat{a}_1 \rangle \langle \hat{a}_2^\dagger \hat{a}_2 \rangle} \tag{7.89}$$

was computed (Fig. 7.7).[8]

7) It should be noted that knowledge of $p(x,\varphi)$ at all phases within a π interval is not necessary to reconstruct a chosen $\langle \hat{a}^{\dagger m} \hat{a}^n \rangle$ and therefore the φ integral in Eq. (7.84) can be replaced by a sum. It was shown that $\langle \hat{a}^{\dagger m} \hat{a}^n \rangle$ can already be obtained from $p(x,\varphi)$ at $N = m+n+1$ different phases φ_k.

8) Here, $\hat{a}_1$ and $\hat{a}_2$ are the photon destruction operators of the non-monochromatic modes defined by the local-oscillator pulses centered at times t_1 and t_2.

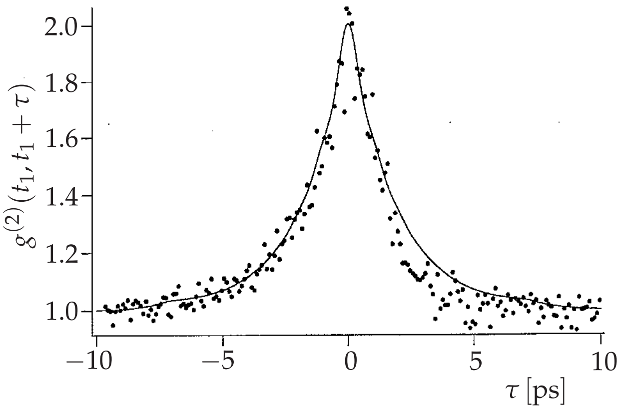

Fig. 7.7 The second-order coherence, Eq. (7.89), experimentally determined via balanced four-port homodyne detection (dots) and from the measured optical spectrum (solid line). The value of t_1 is set to occur near the maximum of the signal pulse. [After McAlister and Raymer (1997).]

Let $\hat{F}$ be an operator whose expectation value $\langle\hat{F}\rangle$ possesses the normally-ordered moment expansion

$$\langle\hat{F}\rangle = \sum_{mn} c_{mn}\langle\hat{a}^{\dagger m}\hat{a}^{n}\rangle. \tag{7.90}$$

In order to relate $\langle\hat{F}\rangle$ to $p(x,\varphi)$, according to the sampling formula (7.73), the kernel function $K_F(x,\varphi)$ may be calculated by substituting for $\langle\hat{a}^{\dagger m}\hat{a}^{n}\rangle$ in Eq. (7.90) the integral representation (7.84) [together with Eqs (7.85) and (7.86)], i. e.,

$$K_F(x,\varphi) = \sum_{m,n} c_{mn}M_{mn}(x)e^{i(m-n)}. \tag{7.91}$$

Recall that $K_F(x,\varphi)$ is only determined up to a function $\Theta(x,\varphi)$ that satisfies the integral equation (7.74).

We identify the operator $\hat{F}$ with the flip operator

$$\hat{A}_{nm} = |n\rangle\langle m| \tag{7.92}$$

in the number basis. It can easily be proved correct, by using Eq. (3.59) and calculating the c-number function $A_{nm}(\alpha;1) = \langle\alpha|n\rangle\langle m|\alpha\rangle$ of $\hat{A}_{nm}$ in normal order, that $\hat{A}_{nm}$ can be written as[9]

$$\hat{A}_{nm} = \frac{1}{\sqrt{n!m!}}\; :\hat{a}^{\dagger n}\hat{a}^{m}e^{-\hat{n}}: . \tag{7.93}$$

9) Note that for $m=n$ Eq. (7.93) reduces to Eq. (4.60).

Equation (7.93) implies that the density-matrix elements in the number basis can be given in the form of the series (7.90),

$$\varrho_{mn} = \langle \hat{A}_{nm} \rangle = \sum_l \frac{(-1)^l}{l!\sqrt{n!m!}} \langle \hat{a}^{\dagger n+l} \hat{a}^{m+l} \rangle, \tag{7.94}$$

which reveals that, when the whole manifold of moments $\langle \hat{a}^{\dagger m} \hat{a}^n \rangle$ is known, then the quantum state is also known in principle.[10] Application of Eq. (7.91) just yields the sampling formula (7.59), where the kernel function is given [up to a function $\Theta(x, \varphi)$] by

$$K_{mn}(x, \varphi) = e^{-i(m-n)\varphi} \sum_l \frac{(-1)^l}{l!\sqrt{n!m!}} M_{n+l\,m+l}(x). \tag{7.95}$$

7.6
Canonical phase statistics

Direct sampling of quantities from the phase-rotated quadrature statistics may be used advantageously when no other method for direct detection of the quantities is available. A typical example is the canonical phase. Whereas the photon number can be measured by direct photodetection, in principle, there has been no apparatus for direct detection of the canonical phase. Therefore, the question arises as to whether or not the phase statistics can be sampled from the phase-rotated quadrature statistics.

Let Ψ_k be the exponential phase moments, i. e., the Fourier components of the phase distribution $p(\phi)$,

$$p(\phi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-ik\phi} \Psi_k\,, \tag{7.96}$$

$$\Psi_k = \int_{2\pi} \mathrm{d}\phi\, e^{ik\phi} p(\phi), \tag{7.97}$$

where $p(\phi) = \langle \phi | \hat{\varrho} | \phi \rangle$, with $|\phi\rangle$ being given by Eq. (3.237). It is not difficult to prove that the substitution of this expression into Eq. (7.97) yields

$$\Psi_k = \sum_n \varrho_{n+k\,n} = \langle \hat{V}^k \rangle \tag{7.98}$$

if $k \geq 0$, and $\Psi_{-k} = \Psi_k^*$ if $k < 0$, with the operator $\hat{V}$ being defined in Eq. (3.220). The sampling function for $\varrho_{n+k\,n}$ can be taken from Eq. (7.95) for $m = n+k$. By

10) Since the moments are not necessarily bounded, the expansion of the density-matrix elements according to Eq. (7.94) does not necessarily converge. The problem of non-convergence may be overcome by analytic continuation of appropriately chosen generating functions [for details, see Herzog (1996)].

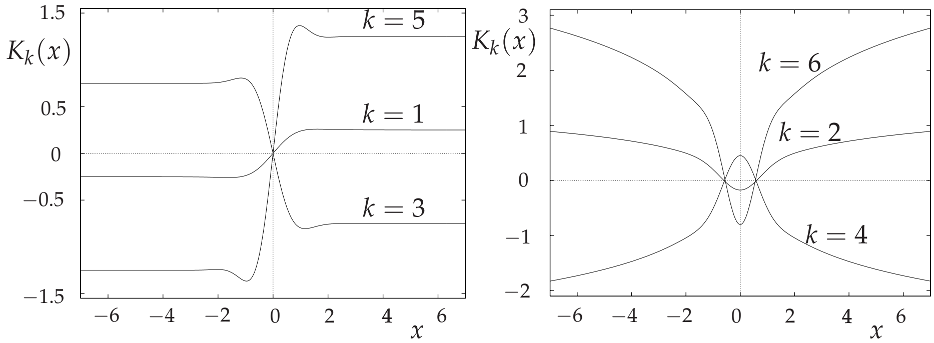

Fig. 7.8 Examples of the kernel function $f_k(x)=2K_k(x/\sqrt{2})$ for sampling the exponential moments of the canonical phase from the phase-rotated quadrature distributions according to Eq. (7.99). [After Dakna, Opatrný and Welsch (1998).]

summing the result over n, we then obtain the sampling function for the kth exponential phase moment. Thus, we arrive at the result that [Dakna, Opatrný and Welsch (1998)]

$$\Psi_k = \int_\pi \mathrm{d}\varphi \int \mathrm{d}x\, e^{-ik\varphi} f_k(x) p(x,\varphi), \tag{7.99}$$

where

$$f_k(x) = \sum_{n,l} \frac{(-1)^l}{l!\sqrt{n!(n+k)!}} M_{n+l\,n+l+k}(x) + \Theta_k(x). \tag{7.100}$$

According to Eq. (7.74), we have introduced a function $\Theta_k(x)$ in order to take into account the ambiguity of the kernel function. It should be chosen so that $f_k(x)$ takes the simplest possible form best suited for sampling Ψ_k from $p(x,\varphi)$. In particular, $\Theta_k(x)$ can be an arbitrary polynomial of degree $k'=k-2n$ (n, integer).

To obtain insight into the structure of the kernel function, we note that Eq. (7.99) applies to quantum and classical systems in a unified way and bridges the gap between quantum and classical phase. It is not difficult to prove that Eq. (7.35) can be rewritten as

$$p(x,\varphi) = \int \mathrm{d}^2\alpha\, W(\alpha)\delta[x - 2|\alpha|\cos(\varphi + \phi_\alpha)]. \tag{7.101}$$

Substitution of this expression into Eq. (7.99) yields ($\mathrm{d}^2\alpha = r\mathrm{d}r\,\mathrm{d}\phi$, $\phi_\alpha \mapsto \phi$)

$$\Psi_k = \int_\pi \mathrm{d}\varphi \int_0^\infty r\mathrm{d}r \int_{2\pi} \mathrm{d}\phi\, e^{-ik(\varphi-\phi)} f_k(2r\cos\varphi) W\big(re^{i\phi}\big). \tag{7.102}$$

Classically, the phase distribution is simply given by the radially integrated Wigner function (i. e., the radially integrated classical phase-space probability

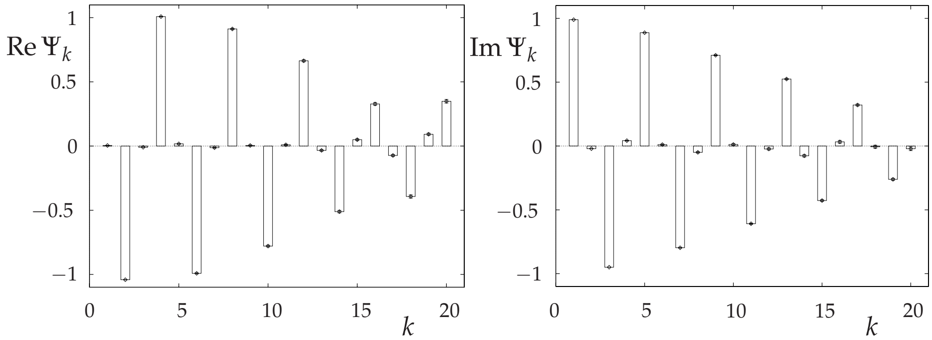

Fig. 7.9 From the phase-rotated quadrature distributions according to Eq. (7.99) reconstructed exponential phase moments of a phase-squeezed state. [After Dakna, Breitenbach, Mlynek, Opatrný, Schiller and Welsch (1998).]

distribution function),

$$p(\phi) = \int_0^\infty r\mathrm{d}r\, W\big(re^{i\phi}\big), \tag{7.103}$$

so that from Eq. (7.97) it follows that

$$\Psi_k = \int_{2\pi} \mathrm{d}\phi \int_0^\infty r\mathrm{d}r\, e^{ik\phi} W\big(re^{i\phi}\big). \tag{7.104}$$

We compare Eq. (7.104) with (7.102) and find that the classical kernel function $f_k(x,\varphi)$ observed for $|x| \to \infty$ satisfies the integral equation

$$\int_\pi \mathrm{d}\varphi\, e^{-ik\varphi} f_k(2r\cos\varphi) = 1 \tag{7.105}$$

for all r. As can be verified by direct substitution, a solution of Eq. (7.105) is ($k > 0$)

$$f_k(x) = \begin{cases} \frac{1}{2}(-1)^{(k-1)/2} k\, \mathrm{sign}\, x & \text{if } k \text{ odd}, \\ \pi^{-1}(-1)^{(k+2)/2} k \ln(x/\sqrt{2}) & \text{if } k \text{ even}. \end{cases} \tag{7.106}$$

Already from the classical kernel (7.106) it can be seen that $\sum_{k=-\infty}^{+\infty} e^{ik(\varphi-\phi)} f_k(x)$ does not exist, and hence the canonical phase distribution $p(\phi)$ itself cannot be sampled from $p(x,\varphi)$ without knowledge of the state.

Choosing $\Theta_k(x)$ in Eq. (7.100) such that $f_k(x)$ approaches the classical limit in the form given by Eq. (7.105) ensures that Eq. (7.99) is suitable for statistical sampling of the exponential phase moments from the phase-rotated quadrature distributions. Examples of $f_k(x)$ are shown in Fig. 7.8. The figure reveals that $f_k(x)$ rapidly approaches the classical limit [Eq. (7.106)] and differs from it

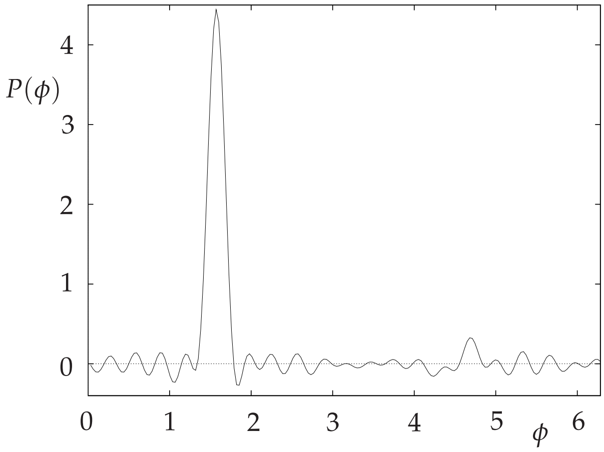

Fig. 7.10 The canonical phase distribution calculated from 20 reconstructed exponential phase moments as shown in Fig. 7.9. [After Dakna, Breitenbach, Mlynek, Opatrný, Schiller and Welsch (1998).]

only in a small interval around the origin, which defines the quantum regime. Obviously, the extension of the interval is just of the order of magnitude of the vacuum fluctuation. Recall that the quantum regime is realized when the state under study has a substantial overlap with the vacuum state.

Figure 7.9 shows the application of the method to the experimental determination of the exponential phase moments of phase-squeezed light [Dakna, Breitenbach, Mlynek, Opatrný, Schiller and Welsch (1998)]. The canonical phase distribution can then be obtained by straightforward summation according to Eq. (7.96), as shown in Fig. 7.10. It should be pointed out that sampling of the first exponential phase moment and the photon-number variance is already sufficient to verify fundamental phase-number uncertainties. The method also applies to the determination of the cosine- and sine-phase statistics associated with the Hermitian operators $\hat{C}$ [Eq. (3.246)] and $\hat{S}$ [Eq. (3.247)]. In particular, the mean values of the cosine phase and the sine phase, respectively, are simply given by the first exponential phase moment according to the relations

$$\langle \hat{C} \rangle = \tfrac{1}{2}(\Psi_1 + \Psi_1^*) \tag{7.107}$$

and

$$\langle \hat{S} \rangle = \tfrac{1}{2i}(\Psi_1 - \Psi_1^*). \tag{7.108}$$

Generally, the moments $\langle \hat{C}^n \rangle$ and $\langle \hat{S}^n \rangle$ ($n \geq 2$) can be obtained from the exponential phase moments and vacuum-assisted density-matrix elements.[11] Finally, the method can also be extended to multi-mode fields. In particular, sampling of the two-mode exponential moments

$$\Psi_{12}^{kl} = \langle \hat{V}_1^k \hat{V}_2^l \rangle \tag{7.109}$$

may be suitable for determining the difference-phase statistics of the modes.

References

Banaszek, K. and K. Wódkiewicz (1996) *Phys. Rev. Lett.* **76**, 4344.

Banaszek, K., C. Radzewicz, K. Wódkiewicz and J.S. Krasiński (1999) *Phys. Rev. A* **60**, 674.

Dakna, M., T. Opatrný and D.-G. Welsch (1998) *Opt. Commun.* **148**, 355.

Dakna, M., G. Breitenbach, J. Mlynek, T. Opatrný, S. Schiller and D.-G. Welsch (1998) *Opt. Commun.* **152**, 289.

D'Ariano, G.M., U. Leonhardt and H. Paul (1995) *Phys. Rev. A* **52**, R1801.

Fano, U. (1957) *Rev. Mod. Phys.* **29**, 74.

Herzog, U. (1996) *Phys. Rev. A* **53**, 2889.

Kis, Z., T. Kiss, J. Janszky, P. Adam, S. Wallentowitz and W. Vogel (1999) *Phys. Rev. A* **59**, R39.

Kiss, T., U. Herzog and U. Leonhardt (1995) *Phys. Rev. A* **52**, 2433.

Kühn, H., D.-G. Welsch and W. Vogel (1994) *J. Mod. Opt.* **41**, 1607.

Lambrecht, A., T. Coudreau, A.M. Steinberg and E. Giacobino (1996) *Europhys. Lett.* **36**, 93.

Leibfried, D., D.M. Meekhof, B.E. King, C. Monroe, W.M. Itano and D.J. Wineland (1996) *Phys. Rev. Lett.* **77**, 4281 (1996).

Leonhardt, U., M. Munroe, T. Kiss, Th. Richter and M.G. Raymer (1996) *Opt. Commun.* **127**, 144.

Leonhardt, U. (1997) *Measuring the Quantum State of Light* (Cambridge University Press).

Mancini, S., V.I. Man'ko and P. Tombesi (1997) *J. Mod. Opt.* **44**, 2281.

McAlister, D.F., and M.G. Raymer (1997) *Phys. Rev. A* **55**, R1609.

von Neumann, J. (1932) *Mathematische Grundlagen der Quantenmechanik* (Springer-Verlag, Berlin).

Opatrný, T., and D.-G. Welsch (1997) *Phys. Rev. A* **55**, 1462.

Richter, Th. (1996a) *Phys. Lett. A* **211**, 327.

Richter, Th. (1996b) *Phys. Rev. A* **53**, 1197.

Schiller, S., G. Breitenbach, S.F. Pereira, T. Müller and J. Mlynek (1996) *Phys. Rev. Lett.* **77**, 2933.

Smithey, D.T., M. Beck, M.G. Raymer and A. Faridani (1993) *Phys. Rev. Lett.* **70**, 1244.

Vogel, K. and H. Risken (1989) *Phys. Rev. A* **40**, 2847.

Wallentowitz, S. and W. Vogel (1996) *Phys. Rev. A* **53**, 4528.

Welsch, D.-G., W. Vogel and T. Opatrný (1999) in *Progress in Optics*, Vol. XXXIX, ed. E. Wolf (Elsevier, Amsterdam), p. 63.

Wigner, E.P. (1932) *Phys. Rev.* **40**, 749.

Zucchetti, A., W. Vogel, M. Tasche and D.–G. Welsch (1996) Phys. Rev. A **54**, 1678.

11) The kernel functions for sampling the exponential phase moments and the vacuum-assisted density-matrix elements can of course be combined to kernel functions that are directly related to $\langle \hat{C}^n \rangle$ and $\langle \hat{S}^n \rangle$.